Turbulence-Induced 2D Correlated Image Distortion

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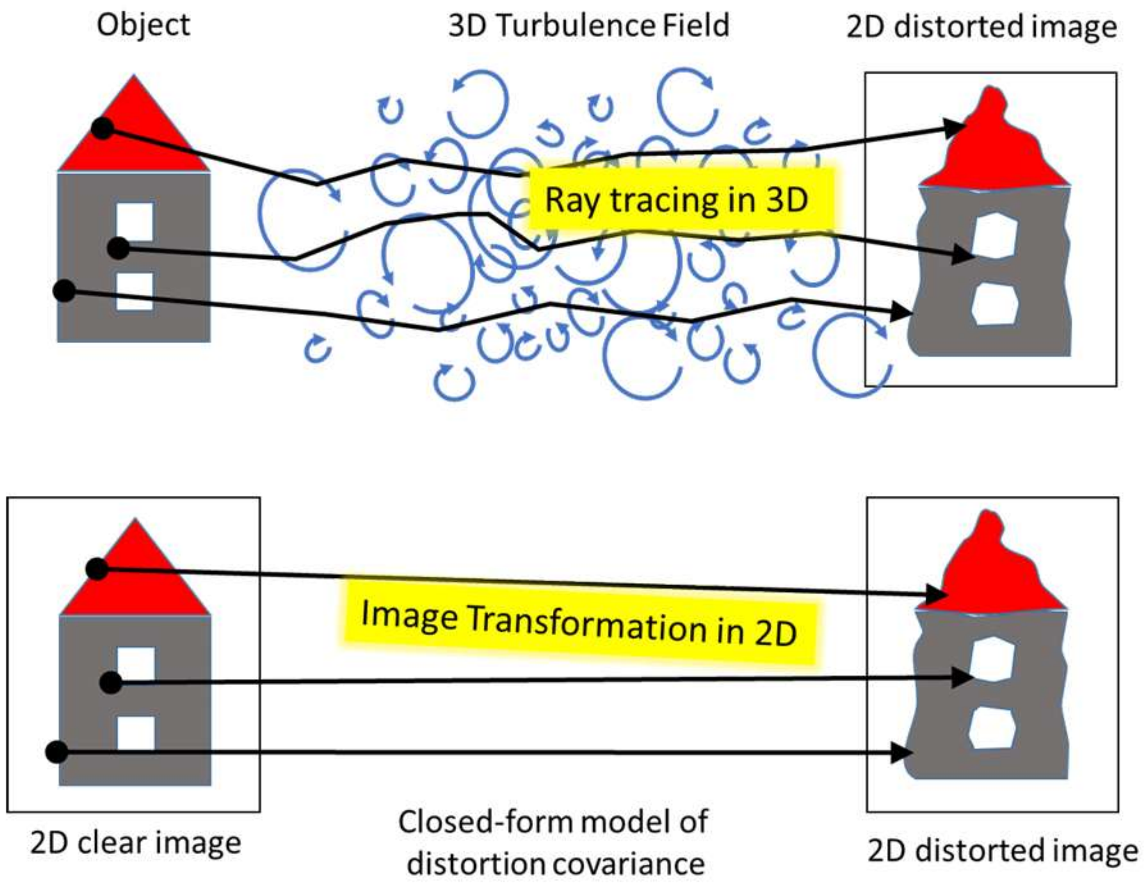
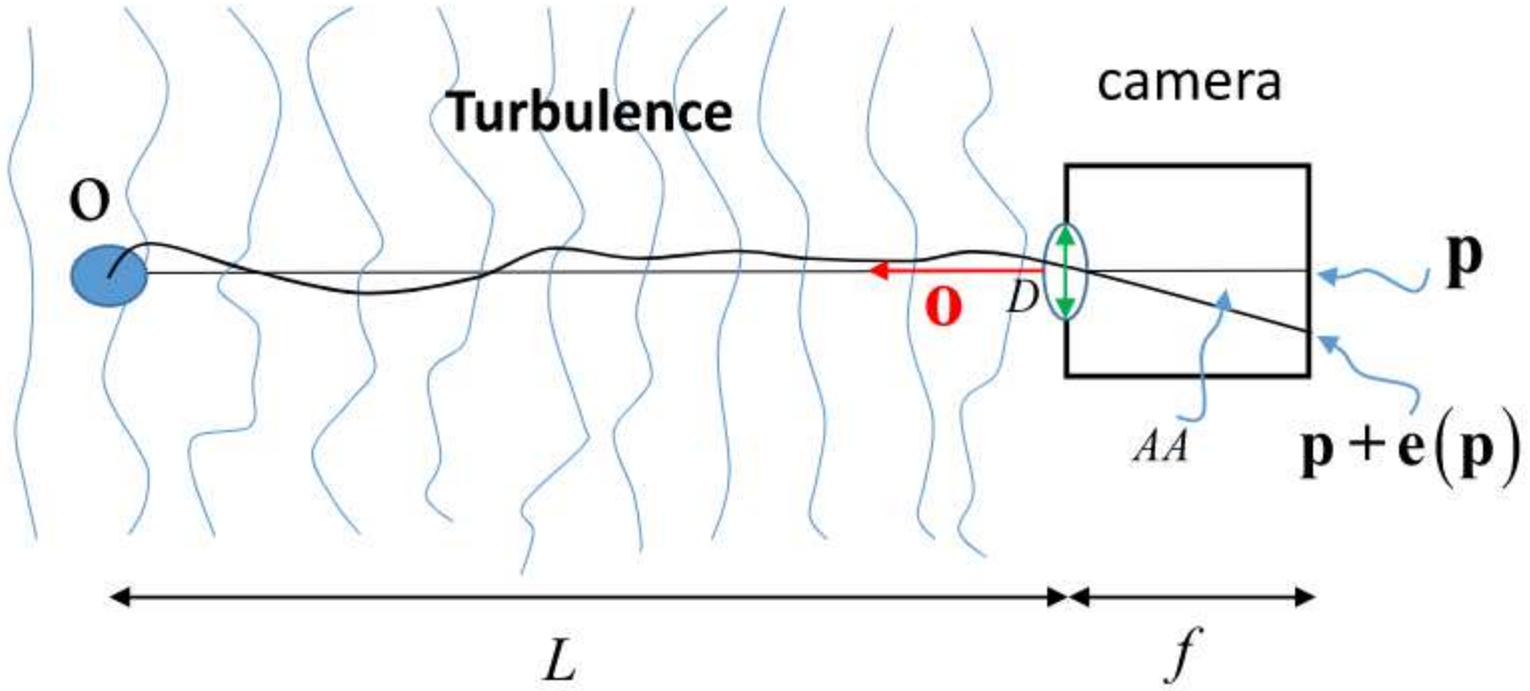


Figure 1: [Top] Computationally complex rendering of image distortion, requiring a 3D simulated random turbulent field, then ray tracing in 3D for all object points in the field of view. [Bottom] Our approach. Efficient rendering of turbulence-induced distortion as 2D image processing. It relies on closed-form physics-based covariance of the distortion.



$$\sigma^2_{\mathrm{AA}} = \gamma(q) C_n^2 L D^{-1/3}, \quad q = \frac{D}{\sqrt{\lambda L}}. \qquad \gamma(q) \approx 1.09$$

Y. Cheon and A. Muschinski. Closed-form approximations for the angle-of-arrival variance of plane and spherical waves propagating through homogeneous and isotropic turbulence. JOSA A, 24(2):415–422, 2007.

相关性公式推导

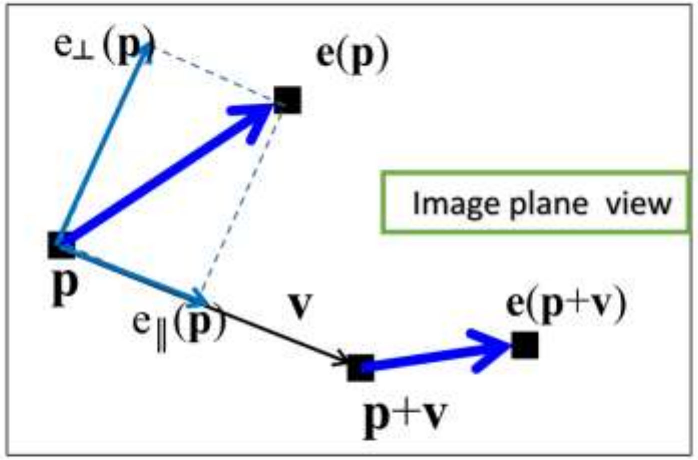


Figure 5: The image plane. In absence of distortion, two object points project to $\mathbf{p}$ and $\mathbf{p}+\mathbf{v}$. Turbulence induces projection displacements, $\mathbf{e}(\mathbf{p})$ and $\mathbf{e}(\mathbf{p}+\mathbf{v})$, respectively. Any displacement vector $\mathbf{e}(\mathbf{p})$ can be divided to components parallel and perpendicular to $\mathbf{v}$. These components are denoted $e_{\|}(\mathbf{p}), e_{\perp}(\mathbf{p})$.

由于湍流的空间相关性，我们更加关注p点和p+v点之中的相关性，其中p点处的位移场用e(p)表示

$$\mathbf{C}(\mathbf{v}) = E\left[\mathbf{e}(\mathbf{p})\mathbf{e}^{\mathrm{T}}(\mathbf{p}+\mathbf{v})\right]. \tag{18}$$

首先定义单位向量vˆ：

$$\hat{\mathbf{v}} = \mathbf{v}/r. \tag{19}$$

从图 5 中，位移矢量场 e（p） 的纵向分量是 e（p） 在 vˆ 上的投影：

$$e_{\|}(\mathbf{p}) = \langle\hat{\mathbf{v}}, \mathbf{e}(\mathbf{p})\rangle = \hat{\mathbf{v}}^{\mathrm{T}}\mathbf{e}(\mathbf{p}), \tag{20}$$

同时设置一个旋转矩阵以表示垂直分量：

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{21}$$

$$\hat{\mathbf{v}}^{\mathrm{T}}\mathbf{R}\hat{\mathbf{v}} = 0. \tag{22}$$

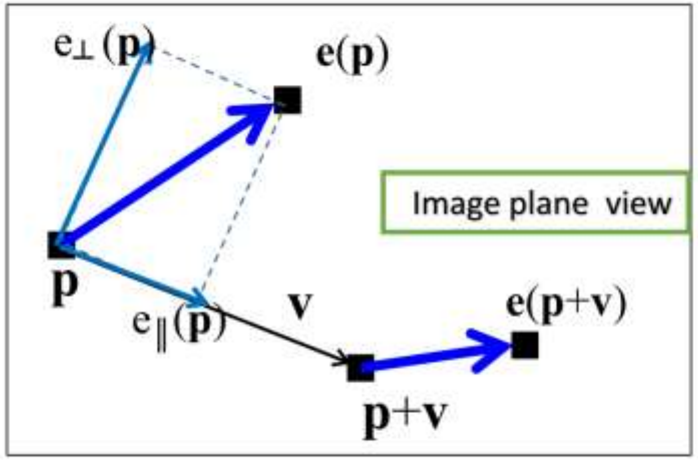
$$e_{\perp}(\mathbf{p}) = \langle\mathbf{R}\hat{\mathbf{v}}, \mathbf{e}(\mathbf{p})\rangle = \hat{\mathbf{v}}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{e}(\mathbf{p}). \tag{23}$$

将公式20和23带入公式18中可以得到：

$$C_{\|}(\mathbf{v}) = E\left[e_{\|}(\mathbf{p})e_{\|}^{\mathrm{T}}(\mathbf{p}+\mathbf{v})\right] = E\left[\hat{\mathbf{v}}^{\mathrm{T}}\mathbf{e}(\mathbf{p})\,\mathbf{e}^{\mathrm{T}}(\mathbf{p}+\mathbf{v})\hat{\mathbf{v}}\right] = \hat{\mathbf{v}}^{\mathrm{T}}\mathbf{C}(\mathbf{v})\hat{\mathbf{v}} \tag{24}$$

$$C_{\perp}(\mathbf{v}) = E\left[e_{\perp}(\mathbf{p})e_{\perp}^{\mathrm{T}}(\mathbf{p}+\mathbf{v})\right] = \hat{\mathbf{v}}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{C}(\mathbf{v})\mathbf{R}\hat{\mathbf{v}}. \tag{25}$$

相关性公式推导



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紧跟上式24和25，我们需要找到一个方法以简化问题，将矩阵自相关性分解为平行分量和垂直分量，设置以下函数：

$$\mathbf{C}(\mathbf{v}) = A(\mathbf{v})\mathbf{I} - B(\mathbf{v})\hat{\mathbf{v}}\hat{\mathbf{v}}^{\mathrm{T}}, \qquad (26)$$

为得到上式，进行以下推导，首先考虑任何一个随机向量均可以被划分为平行和垂直两个分量：

$$\vec{w} = \vec{w}_{\parallel} + \vec{w}_{\perp}$$

平行于vˆ 和垂直于vˆ 的分量分别可表示为：

$$\vec{w}_{\parallel} = (\hat{v}\hat{v}^T)\vec{w} \qquad \vec{w}_{\perp} = (R\hat{v}(R\hat{v})^T)\vec{w}$$

由于任何向量都可以如此分解，因此可以得到：

$$\vec{w} = (\hat{v}\hat{v}^T)\vec{w} + (R\hat{v}(R\hat{v})^T)\vec{w} = (\hat{v}\hat{v}^T + R\hat{v}(R\hat{v})^T)\vec{w}$$

所以可以得到：

$$\hat{v}\hat{v}^T + R\hat{v}(R\hat{v})^T = I$$

那么可以得到一个思路以表示C(v)：

$$C(v) = \alpha(v)\hat{v}\hat{v}^T + \beta(v)R\hat{v}(R\hat{v})^T$$

又因为：

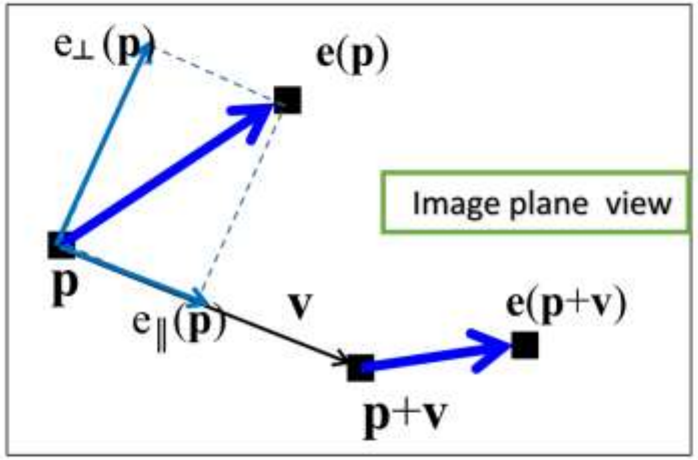
$$R\hat{v}(R\hat{v})^T = I - \hat{v}\hat{v}^T$$

因此综合上述公式可以得到最终结果，如式26所示：

$$C(v) = \alpha(v)\hat{v}\hat{v}^T + \beta(v)(I - \hat{v}\hat{v}^T) = \beta(v)I + (\alpha(v) - \beta(v))\hat{v}\hat{v}^T$$

$$A(v) = \beta(v)\text{和}B(v) = \beta(v) - \alpha(v)$$

相关性公式推导



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承接上页中式26，将其用于式24和25，可以得到：

$$C_{\|}(\mathbf{v}) = A(\mathbf{v})\hat{\mathbf{v}}^{T}\hat{\mathbf{v}} - B(\mathbf{v})(\hat{\mathbf{v}}^{T}\hat{\mathbf{v}})^{2} = A(\mathbf{v}) - B(\mathbf{v}) \quad (28)$$

$$C_{\perp}(\mathbf{v}) = A(\mathbf{v})\hat{\mathbf{v}}^{T}\mathbf{R}^{T}\mathbf{R}\hat{\mathbf{v}} - B(\mathbf{v})\hat{\mathbf{v}}^{T}\mathbf{R}^{T}\hat{\mathbf{v}}\hat{\mathbf{v}}^{T}\mathbf{R}\hat{\mathbf{v}} = A(\mathbf{v}). \quad (29)$$

将式28和29反带入式26中可得：

$$\mathbf{C}(\mathbf{v}) = C_{\perp}(\mathbf{v})\mathbf{I} - \left[C_{\perp}(\mathbf{v}) - C_{\|}(\mathbf{v})\right]\hat{\mathbf{v}}\hat{\mathbf{v}}^{T} \ . \quad (30)$$

相关性公式与物理属性结合

从物体的角度来看，相机孔径所占据的角尺寸可以用下式表示：

$$\theta_D = \frac{D}{L}. \tag{9}$$

两个物体之间的角度和孔径的角尺寸之间若用归一化方法评估可以使用下式：

$$\eta = \frac{\theta}{\theta_D}. \tag{10}$$

湍流外尺度与相机镜头直径归一化可以表示为：

$$\rho = \frac{L_0}{D}. \tag{11}$$

对于从镜头到物体之间这段距离而言可以定义几个辅助函数以表达湍流的传播过程：

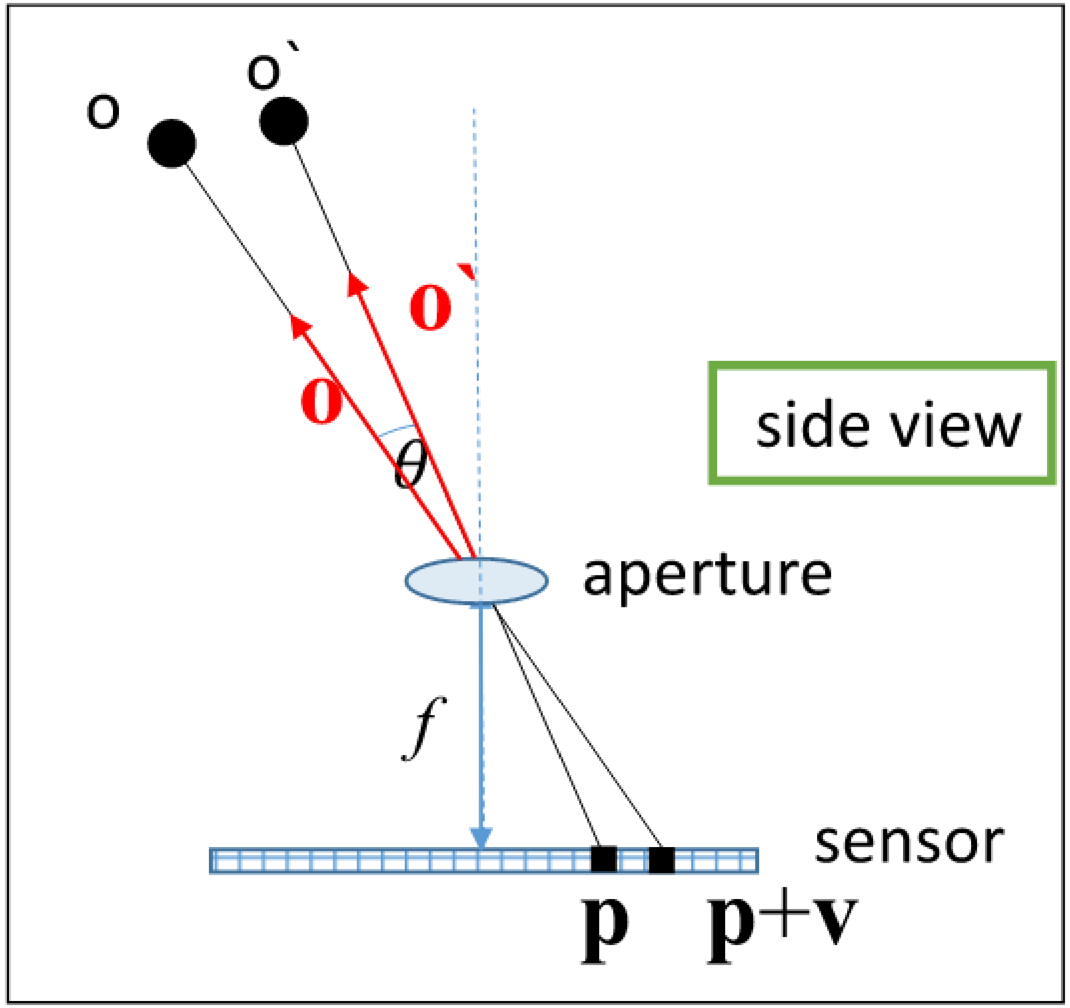
$$Q(\xi) = (1-\xi)^{5/3} \quad , \qquad M_\eta = 2\eta\xi(1-\xi)\ . \tag{12}$$

$$H(\kappa) = \frac{[(2\rho\kappa)^2+1]^{-11/6} J_2^2(\kappa)}{\kappa}. \tag{14}$$

对于真实物理参数而言，平行分量和垂直分量则可以使用下式定义：

$$b_{\parallel,\perp}(\eta) = \frac{\int_0^1 \left\{ \int_0^\infty H(\kappa)[J_0(M_\eta\kappa) \mp J_2(M_\eta\kappa)]d\kappa \right\} Q(\xi)d\xi}{\int_0^1 \left\{ \int_0^\infty H(\kappa)d\kappa \right\} Q(\xi)d\xi} \tag{15}$$

相关性公式与物理属性结合



从左图中可以看出，对于两个物体之间的角度可以用下式来表示：

$$\theta \approx \frac{|\mathbf{v}|}{f} = \frac{r}{f}, \tag{16}$$

那么结合式10和15可以表达为：

$$b_{\|,\perp}(\eta) = b_{\|,\perp}\left(\frac{r}{f\theta_D}\right). \tag{17}$$

回到式28和29得到：

$$C_{\|}(\mathbf{v}) = b_{\|}\left(\frac{r}{f\theta_D}\right) \quad , \quad C_{\perp}(\mathbf{v}) = b_{\perp}\left(\frac{r}{f\theta_D}\right). \tag{31}$$

式26可最终表示为：

$$\mathbf{C}(\mathbf{v}) = A(r)\mathbf{I} - B(r)\hat{\mathbf{v}}\hat{\mathbf{v}}^{\mathrm{T}} \tag{32}$$

$$\begin{aligned} A(r) &= b_{\perp}\left(\frac{r}{f\theta_D}\right) \\ B(r) &= b_{\perp}\left(\frac{r}{f\theta_D}\right) - b_{\|}\left(\frac{r}{f\theta_D}\right). \end{aligned} \tag{33}$$

创建扭曲场

本部分目的是生成符合物理逻辑的随机扭曲场e(p)，设 z（p） 为白噪声向量场，其中K(p)为矩阵值卷积核：

$$\mathbf{e}(\mathbf{p}) = \int \mathbf{K}(\mathbf{p}-\mathbf{p}')\mathbf{z}(\mathbf{p}')d\mathbf{p}'. \qquad (34)$$

通过上式与式18结合我们可以推导出下式：

$$\begin{aligned} C(v) &= \mathbb{E}[e(p)e^T(p+v)] \\ &= \mathbb{E}\left[\int K(p-p_1)z(p_1)dp_1 \cdot \int z^T(p_2)K^T(p+v-p_2)dp_2\right] \\ &= \iint K(p-p_1)\mathbb{E}[z(p_1)z^T(p_2)]K^T(p+v-p_2)dp_1dp_2 \\ &= \int K(p-p_1)K^T(p+v-p_1)dp_1 \\ &= \int K(u)K^T(u+v)du \end{aligned}$$

将上述两式进行频域转化得到：

$$\tilde{\mathbf{e}}(\boldsymbol{\omega}) = \int \mathbf{e}(\mathbf{p})e^{-j\boldsymbol{\omega}^{\mathsf{T}}\mathbf{p}}d\mathbf{p} = \tilde{\mathbf{K}}(\boldsymbol{\omega})\tilde{\mathbf{z}}(\boldsymbol{\omega}). \qquad (36)$$

$$\tilde{\mathbf{C}}(\boldsymbol{\omega}) = \int \mathbf{C}(\mathbf{v})e^{-j\boldsymbol{\omega}^{\mathsf{T}}\mathbf{v}}d\mathbf{v} = E\left[\tilde{\mathbf{e}}^*(\boldsymbol{\omega})\tilde{\mathbf{e}}^{\mathsf{T}}(\boldsymbol{\omega})\right], \qquad (37)$$

此外，如果将式36带入到式37中可以推导出：

$$\begin{aligned} \tilde{\mathbf{C}}(\boldsymbol{\omega}) &= E\left[\tilde{\mathbf{K}}^*(\boldsymbol{\omega})\tilde{\mathbf{z}}^*(\boldsymbol{\omega})\tilde{\mathbf{z}}^{\mathsf{T}}(\boldsymbol{\omega})\tilde{\mathbf{K}}^{\mathsf{T}}(\boldsymbol{\omega})\right] \\ &= \tilde{\mathbf{K}}^*(\boldsymbol{\omega})E\left[\tilde{\mathbf{z}}^*(\boldsymbol{\omega})\tilde{\mathbf{z}}^{\mathsf{T}}(\boldsymbol{\omega})\right]\tilde{\mathbf{K}}^{\mathsf{T}}(\boldsymbol{\omega}) = \tilde{\mathbf{K}}^*(\boldsymbol{\omega})\tilde{\mathbf{K}}^{\mathsf{T}}(\boldsymbol{\omega}). \end{aligned} \qquad (38)$$

创建扭曲场

为得到 $\tilde{\mathbf{K}}(\omega)$ 与 $\tilde{\mathbf{C}}(\omega)$ 的关系式，同时能够满足式38，可以使用下列公式：

$$\tilde{\mathbf{K}}(\omega) = \frac{1}{t(\omega)}\left[\tilde{\mathbf{C}}(\omega) + s(\omega)\mathbf{I}\right] \quad . \qquad (39)$$

$$s(\omega) = \sqrt{\det \tilde{\mathbf{C}}(\omega)}$$

$$t(\omega) = \sqrt{\mathrm{tr}\tilde{\mathbf{C}}(\omega) + 2s(\omega)}.$$

最终整体的扭曲过程可以如下表示：

1. 首先，我们使用公式32和33计算自相关函数$C(v)$
2. 对$C(v)$进行傅里叶变换得到$\tilde{C}(\omega)$
3. 使用公式39计算$\tilde{K}(\omega)$
4. 生成随机白噪声场$z(p)$并进行傅里叶变换得到$\tilde{z}(\omega)$
5. 在频域中乘以$\tilde{K}(\omega)$和$\tilde{z}(\omega)$得到$\tilde{e}(\omega)$
6. 对$\tilde{e}(\omega)$进行逆傅里叶变换，得到位移场$e(p)$
7. 根据公式1调整$e(p)$的方差

B. W. Levinger. The square root of a 2× 2 matrix. Mathematics Magazine, 53(4):222–224, 1980.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Turbulence induced distortion
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